

# A LIOUVILLE THEOREM FOR HIGH ORDER DEGENERATE ELLIPTIC EQUATIONS

GENGGENG HUANG AND CONGMING LI

ABSTRACT. In this paper, we apply the moving plane method to the following high order degenerate elliptic equation,

$$(-A)^p u = u^\alpha \text{ in } \mathbb{R}_+^{n+1}, n \geq 1,$$

where the operator  $A = y\partial_y^2 + a\partial_y + \Delta_x, a \geq 1$ . We get a Liouville theorem for subcritical case and classify the solutions for the critical case.

Key Words: Degenerate elliptic, Moving plane, Divergence identity

## 1. INTRODUCTION

This article concerns the symmetry of solutions of degenerate elliptic equations on an unbounded domain. The first well-known work was first done by Gidas, Ni and Nirenberg [6] for the uniformly elliptic equations. In the elegant paper of [6], one of the interesting results is on the symmetries of the non-negative solutions of

$$(1.1) \quad \Delta u + u^\alpha = 0, \quad x \in \mathbb{R}^n, n \geq 3.$$

They classified the positive solutions of (1.1) for  $\alpha = \frac{n+2}{n-2}$  with additional decay at infinity, namely  $u(x) = O(|x|^{2-n})$  by the method of moving plane. Later on, Caffarelli, Gidas and Spruck [2] removed the growth assumption by introducing the Kelvin transformation and got the same results. In the case that  $1 \leq \alpha < \frac{n+2}{n-2}$ , Gidas and Spruck [7] showed that (1.1) admitted only trivial solution.

An interesting related problem is the extension of (1.1) to the degenerate elliptic case,

$$(1.2) \quad y\partial_y^2 u + \Delta_x u + a\partial_y u + u^\alpha = 0, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^+ = \mathbb{R}_+^{n+1}, \quad n \geq 1,$$

here  $a \geq 1$  is a constant. Such an equation arises from the isometric embedding of Alexandrov-Nirenberg surfaces when we dealt with the a priori estimates of the second fundamental forms, see [8]. In [9], the author got that for  $1 < \alpha < \frac{n+2a+2}{n+2a-2}$ , the only nonnegative solutions for (1.2) is 0 and classified all the nonnegative solutions for  $\alpha = \frac{n+2a+2}{n+2a-2}$ .

Then, it's natural to consider the high order degenerate elliptic case. we consider the nonnegative solutions  $u \in C^{2p}(\overline{\mathbb{R}_+^{n+1}})$  of the following high order degenerate elliptic equations

$$(1.3) \quad (-A)^p u = u^\alpha \text{ in } \mathbb{R}_+^{n+1}, n \geq 1,$$

where the operator  $A = y\partial_y^2 + \Delta_x + a\partial_y, 1 \leq p < \frac{n+2a}{2}, p \in \mathbb{Z}$  and  $a \geq 1$  is a constant. As is known, there are many high order elliptic extension results concerning (1.1), for instance [3, 11, 12, 13, 15, 16] and references therein. Inspired by these results, we have the following theorem of (1.3).

**Theorem 1.1.** *Let  $0 \leq u(x, y) \in C^{2p}(\overline{\mathbb{R}_+^{n+1}})$  satisfy the following equation,*

$$(1.4) \quad (-A)^p u = u^\alpha, \quad \text{in } \mathbb{R}_+^{n+1}, \quad p \in \mathbb{Z}^+, 2p < n + 2a,$$

where the operator  $A = y\partial_y^2 + a\partial_y + \Delta_x$ ,  $a \geq 1$  is a constant. Then

- (1) for  $1 < \alpha < \frac{n+2a+2p}{n+2a-2p}$ ,  $u \equiv 0$ ;
- (2) for  $\alpha = \frac{n+2a+2p}{n+2a-2p}$ ,  $u_{t,x_0}(x, y) = c_0 \left( \frac{t}{t^2 + 4y + |x - x_0|^2} \right)^{\frac{n+2a-2p}{2}}$

for some  $x_0 \in \mathbb{R}^n$  and  $t > 0$ .

We will prove Theorem 1.1 by the method of moving plane. Noting the classification in Theorem 1.1, we think  $x, y$  play different scales in the equation (1.4). So in fact, we always take the transformation  $x_{n+1} = 2\sqrt{y}$  to make (1.4) easier to be dealt with. After the transformation  $x_{n+1} = 2\sqrt{y}$ , (1.4) changes to

$$(1.5) \quad (-\tilde{\Delta}_{n+1,a})^p u = u^\alpha, \quad \text{in } \mathbb{R}_+^{n+1}, \quad p \in \mathbb{Z}^+, 2p < n + 2a,$$

where the operator  $\tilde{\Delta}_{n+1,a} = \sum_{i=1}^{n+1} \partial_{x_i}^2 + \frac{2a-1}{x_{n+1}} \partial_{x_{n+1}}$ .

In order to apply the moving plane method to the high order elliptic cases of (1.1), one important step is to prove that  $(-\Delta)^i u > 0, i = 1, \dots, p-1$ . Similarly, for (1.5), we have

**Theorem 1.2.** *Let  $0 < u(x) \in C^{2p}(\mathbb{R}^{n+1})$  be an even function with respect to  $x_{n+1}$  and satisfy (1.5) in  $\mathbb{R}^{n+1}$ . Then it's valid that*

$$(1.6) \quad (-\tilde{\Delta}_{n+1,a})^i u > 0, \quad \text{in } \mathbb{R}^{n+1}, i = 1, \dots, p-1.$$

In getting Theorem 1.2, we mainly follow the arguments in [15]. Since the appearance of the first order derivative, we can't just use the sphere average  $\bar{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u dS$ , instead, we use weighted average

$$\bar{u}_w(r) = \frac{1}{r^{n+2a-1}} \int_{\partial B_r} |x_{n+1}|^{2a-1} u(x) dS.$$

This is the new idea in the present paper and causes some changes in the proof.

In the proof of Theorem 1.1, we inevitably encounter with the maximal principle for  $u \in C^{2p}(B_1 \setminus \{0\})$  even with respect to  $x_{n+1}$  and

$$(1.7) \quad (-\tilde{\Delta}_{n+1,a})^p u = |x|^{-\tau} u^\alpha, \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^{n+1}, \alpha > 1, \tau = (n+2a+2p) - \alpha(n+2a-2p).$$

In [15], the authors proved that  $|x|^{-\tau} u^\alpha \in L^1(B_1)$ . Then by the maximum principle for weak superharmonic functions, they could show  $(-\Delta)^i u \geq \inf_{\partial B_r} u$ . In our case, we can only prove  $|x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha \in L^1(B_1)$ . And also, we don't have the corresponding maximum principle. Instead of this, we establish a Green formula for (1.7) overcome this difficulty. This idea is originated from [10] for Laplacian equation generalized by [4] for polyharmonic operators. We extend these results to some weighted divergence system. This is also an interesting part of our paper.

**Theorem 1.3.** *Let  $u \in C^{2p}(\bar{B}_1 \setminus \{0\})$  with  $u(x', x_{n+1}) = u(x', -x_{n+1})$  satisfy (1.7). Moreover, we assume that  $|x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha \in L^1(B_1)$ , then*

$$(1.8) \quad \int_{\partial B_1} \left[ \frac{\partial v_i}{\partial r} - (2a-1)v_i \right] dS + \lim_{s \rightarrow 0} \int_{B_1 \setminus B_s} v_{i+1} dx = 0,$$

here  $v_i = |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a})^i u, i = 0, \dots, p-1$ .

The present paper is organized as follows. In Section 2, we will collect some preliminary results concerning about the maximal principles and the asymptotic properties. In Section 3, we will prove the ‘‘superharmonic’’ property of  $(-\tilde{\Delta}_{n+1,a})^i u$  or Theorem 1.2. This section mainly follows

the arguments of [15] except for the utility of weighted spherical average. We will establish a divergence identity in Section 4. This is a generalization of the works of [10] and [4]. Also there are some interesting ideas in both Section 3 and Section 4. The last section is devoted to prove Theorem 1.1.

## 2. PRELIMINARY RESULTS

In this section, we first collect some basic facts.

**Lemma 2.1.** *If  $u(x, y) \in C^{2k}(\overline{\mathbb{R}_+^{n+1}})$  and  $v(x, t) = u(x, \frac{t^2}{4})$ , then we have  $v(x, t) \in C^{2k}(\overline{\mathbb{R}_+^{n+1}})$  and*

$$(2.1) \quad \frac{\partial^{2l-1} v(x, t)}{\partial t^{2l-1}} \Big|_{t=0} = 0, \quad l = 1, 2, \dots, k.$$

*Proof.* It is obvious that  $v(x, t) \in C^{2k}(\overline{\mathbb{R}_+^{n+1}})$ . We only need to show (2.1) is true. For  $l = 1$ , one has  $\frac{\partial v}{\partial t} = \frac{t}{2} \frac{\partial u}{\partial y}$  for  $t = 2\sqrt{y}$ . We prove (2.1) by induction. Suppose for  $l$  we have

$$(2.2) \quad \frac{\partial^{2l-1} v}{\partial t^{2l-1}} = \sum_{k=1}^{2l-1} \sum_{i=1}^l c_{ik,l} t^{2i-1} \frac{\partial^k u}{\partial y^k}, \quad \frac{\partial^{2l} v}{\partial t^{2l}} = \sum_{k=1}^{2l} \sum_{i=0}^l c'_{ik,l} t^{2i} \frac{\partial^k u}{\partial y^k}$$

for some constants  $c_{ik,l}, c'_{ik,l}$ . Then for  $l+1$ , one can see

$$\begin{aligned} \frac{\partial^{2l+1} v}{\partial t^{2l+1}} &= \sum_{k=1}^{2l} \sum_{i=0}^l 2i c'_{ik,l} t^{2i-1} \frac{\partial^k u}{\partial y^k} + \sum_{k=1}^{2l} \sum_{i=0}^l \frac{1}{2} c'_{ik,l} t^{2i+1} \frac{\partial^{k+1} u}{\partial y^{k+1}} = \sum_{k=1}^{2l+1} \sum_{i=1}^{l+1} c_{ik,l+1} t^{2i-1} \frac{\partial^k u}{\partial y^k} \\ \frac{\partial^{2l+2} v}{\partial t^{2l+2}} &= \sum_{k=1}^{2l+1} \sum_{i=1}^{l+1} (2i-1) c_{ik,l+1} t^{2i-2} \frac{\partial^k u}{\partial y^k} + \sum_{k=1}^{2l+1} \sum_{i=1}^{l+1} \frac{1}{2} c_{ik,l+1} t^{2i} \frac{\partial^{k+1} u}{\partial y^{k+1}} = \sum_{k=1}^{2l+2} \sum_{i=0}^{l+1} c'_{ik,l+1} t^{2i} \frac{\partial^k u}{\partial y^k} \end{aligned}$$

This proves (2.2). By (2.2), it is easy to see that  $\frac{\partial^{2l-1} v(x,t)}{\partial t^{2l-1}} \Big|_{t=0} = 0, \quad l = 1, 2, \dots, k. \quad \square$

From Lemma 2.1, we have the following remark.

**Remark 2.1.** *For any  $u(x, y) \in C^{2p}(\overline{\mathbb{R}_+^{n+1}})$ , if we take transformation  $x_{n+1} = 2\sqrt{y}$  and  $\tilde{u}(x, x_{n+1}) = u(x, y)$ . Then we can take the even extension  $\tilde{u}(x, x_{n+1}) = \tilde{u}(x, -x_{n+1})$  such that  $\tilde{u}(x, x_{n+1}) \in C^{2p}(\mathbb{R}^{n+1})$ . This means (1.5) is still true in  $\mathbb{R}^{n+1}$  if we take even extension with respect to  $x_{n+1}$ .*

By Lemma 2.1 and Remark 2.1, we shall always consider (1.5) in  $\mathbb{R}^{n+1}$  by even extension.

In order to use the moving plane method, we need some new maximal principles. Consider the following elliptic operator

$$L(u) = \sum_{i=1}^{n+1} a_{ij}(x) \partial_{ij} u + \sum_{i=1}^n b_i(x) \partial_i u + \frac{a(x)}{x_{n+1}} \partial_{n+1} u.$$

All the coefficients  $a_{ij}(x), b_i(x), a(x) \in C(\mathbb{R}^{n+1})$ ,  $a(x) \geq 0$  and  $(a_{ij})$  is a positive definite matrix. Then we shall have the following two lemmas. Let  $B_1$  be the unit ball centered at origin.

**Lemma 2.2.** *Suppose that  $u \in C^2(B_1) \cap C(\bar{B}_1)$  with  $\partial_{n+1} u(x', 0) = 0$  satisfies that*

$$-L(u) \geq 0 \text{ in } B_1.$$

*Then either  $u$  is a constant or  $u$  can not attain its minimum in  $B_1$ .*

**Lemma 2.3.** Suppose that  $u(x) \in C^2(B_1) \cap C^1(\bar{B}_1)$  with  $\partial_{n+1}u(x', 0) = 0$  satisfies that

$$(2.3) \quad -L(u) \geq 0 \text{ in } B_1.$$

If  $u$  attains its minimum at  $x^0 \in \partial B_1$ , then either  $u \equiv \text{const}$  or

$$-\frac{\partial u}{\partial n}|_{x=x^0} > 0, \text{ } n \text{ is the outward normal to } \partial B_1 \text{ at } x^0.$$

Lemma 2.2 and Lemma 2.3 are obtained in [9], we omit the proof here. Also, we need the following lemma for a punctured ball which was also proved in [9].

**Lemma 2.4.** Suppose that  $v \in C^2(B_1 \setminus \{0\}) \cap C(\bar{B}_1 \setminus \{0\})$  is a solution of the following problem with  $n + 2a > 2$ ,

$$(2.4) \quad \Delta_{n+1}v + \frac{2a-1}{x_{n+1}}\partial_{n+1}v \leq 0, \quad \text{in } B_1 \setminus \{0\}, \quad \partial_{n+1}v(x', 0) = 0.$$

If we have  $\liminf_{x \rightarrow 0} |x|^{n+2a-2}v(x) \geq 0$ , then there holds

$$v(x) \geq \inf_{\partial B_1} v, \quad \forall x \in B_1 \setminus \{0\}.$$

Also we have the following Kelvin transformation,

$$(2.5) \quad u^*(x) = |x|^{2p-n-2a}u\left(\frac{x}{|x|^2}\right).$$

If  $u(x)$  satisfies (1.5), then we have

$$(2.6) \quad (-\tilde{\Delta}_{n+1,a})^p u^* = |x|^{-\tau} (u^*)^\alpha, \quad \text{in } \mathbb{R}^{n+1} \setminus \{0\}, \tau = (n + 2a + 2p) - \alpha(n + 2a - 2p).$$

As the proof of (2.6) is of independent interest, we will present it in the Appendix.

**Lemma 2.5.** Let  $u(x) \in C^\infty(\mathbb{R}^{n+1})$  be an even function with respect to  $x_{n+1}$  and  $u^*(x)$  be defined in (2.5). Then

$$(2.7) \quad (-\tilde{\Delta}_{n+1,a})^i u^*(x) = \frac{c_i}{|x|^{n+2a-2p+2i}} f_i\left(\frac{x}{|x|^2}\right), \quad i = 0, \dots, p-1$$

for some constants  $c_i > 0$  and smooth functions  $f_i(x)$  with  $f_i(0) = u(0)$  and  $f_i(x)$  are even functions with respect to  $x_{n+1}$ . Moreover, we have  $(-\tilde{\Delta}_{n+1,a})^i u^*(x) > 0$  for  $|x|$  large enough.

*Proof.* We shall prove (2.7) by induction. For  $i = 0$ ,  $c_0 = 1$ ,  $f_0(x) = u(x)$ . Now

$$\begin{aligned} (-\tilde{\Delta}_{n+1,a})^{i+1} u^*(x) &= (-\tilde{\Delta}_{n+1,a}) \left( \frac{c_i}{|x|^{n+2a-2p+2i}} f_i\left(\frac{x}{|x|^2}\right) \right) \\ &= \frac{c_i}{|x|^{n+2a-2p+2(i+1)}} \left\{ (2p-2i-2)(n+2a-2p+2i) f_i\left(\frac{x}{|x|^2}\right) \right. \\ &\quad \left. + 4(p-i-1) \sum_{j=1}^{n+1} \frac{x_j}{|x|^2} \frac{\partial f_i}{\partial x_j} \left(\frac{x}{|x|^2}\right) - |x|^{-2} (\tilde{\Delta}_{n+1,a} f_i) \left(\frac{x}{|x|^2}\right) \right\}. \end{aligned}$$

Set

$$\begin{aligned} &(2p-2i-2)(n+2a-2p+2i) f_{i+1}(x) \\ &= (2p-2i-2)(n+2a-2p+2i) f_i(x) + 4(p-i-1) \sum_{j=1}^{n+1} x_j \frac{\partial f_i}{\partial x_j}(x) - |x|^2 \tilde{\Delta}_{n+1,a} f_i(x), \\ &c_{i+1} = (2p-2i-2)(n+2a-2p+2i) c_i \end{aligned}$$

Then

$$(-\tilde{\Delta}_{n+1,a})^{i+1}u^*(x) = \frac{c_{i+1}}{|x|^{n+2a-2p+2(i+1)}}f_{i+1}\left(\frac{x}{|x|^2}\right)$$

and

$$f_{i+1}(0) = f_i(0) = u(0), f_{i+1}(x', x_{n+1}) = f_{i+1}(x', -x_{n+1}).$$

As  $c_i > 0$  for  $i = 0, \dots, p-1$ ,

$$(-\tilde{\Delta}_{n+1,a})^i u^*(x) = \frac{c_i u(0)}{|x|^{n+2a-2p+2i}} \left(1 + O\left(\frac{1}{|x|}\right)\right) > 0$$

for  $|x|$  large enough.  $\square$

### 3. THE “SUPERHARMONIC” PROPERTY

This section is devoted to prove Theorem 1.2.

**The proof of Theorem 1.2:**

Denote  $u_i = (-\tilde{\Delta}_{n+1,a})^i u$ ,  $i = 0, 1, 2, \dots, p-1$  with  $u_0 = u$ . We first prove

$$u_{p-1} > 0.$$

If not, there exists  $x_0 \in \mathbb{R}^{n+1}$  such that

$$u_{p-1}(x_0) < 0.$$

For simplicity we can just assume  $x_0 = 0$ . We need make some explanations here. As  $\tilde{\Delta}_{n+1,a}$  is invariant under the translation of  $x_1, \dots, x_n$ , we can translate  $x_1, \dots, x_n$  to make the first  $n$  coordinates to be 0. Hence we may assume  $x_0 = (0, \dots, 0, b)$  for some  $b \geq 0$ . Now we make a transformation  $\tilde{u}(x', x_{n+1}, x_{n+2}) = u(x', \sqrt{x_{n+1}^2 + x_{n+2}^2})$  where  $x' = (x_1, \dots, x_n)$ . By directly computation, we can derive that

$$(-\tilde{\Delta}_{n+2,a-\frac{1}{2}})^p \tilde{u} = \tilde{u}^\alpha, \quad \text{in } \mathbb{R}^{n+2},$$

where  $-\tilde{\Delta}_{n+2,a-\frac{1}{2}} = \sum_{i=1}^{n+2} \partial_{x_i}^2 + \frac{2a-2}{x_{n+2}} \partial_{x_{n+2}}$ . By the transformation, we have

$$\tilde{u}_{p-1}(x', c, d) = (-\tilde{\Delta}_{n+2,a-\frac{1}{2}})^{p-1} \tilde{u}(x', c, d) = (-\tilde{\Delta}_{n+1,a})^{p-1} u(x', b) = u_{p-1}(x', b), \text{ for } b = \sqrt{c^2 + d^2}.$$

Set  $x' = 0, c = b, d = 0$ , we have for  $\tilde{u}_{p-1}(0, b, 0) < 0$ . Since  $\tilde{u}$  is invariant under translation with respect to  $x_1, \dots, x_{n+1}$ , we have after translation  $\tilde{u}_{p-1}(0) < 0$ . Therefor, by the above arguments, we can just assume  $x_0 = 0$ , otherwise we can consider  $\tilde{u}$ . Set

$$\bar{f}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} f dS.$$

From now on, if no confusion occurs, we will always normalize  $|\mathbb{S}^n| = 1$  where  $\mathbb{S}^n$  is the unit sphere in  $\mathbb{R}^{n+1}$ . Then for  $u_{p-1}$ , we have

$$\begin{aligned} - \int_{B_r} |x_{n+1}|^{2a-1} u^\alpha dx &= \int_{B_r} \nabla \cdot (|x_{n+1}|^{2a-1} \nabla u_{p-1}) dx \\ &= \int_{\partial B_r} |x_{n+1}|^{2a-1} \frac{\partial u_{p-1}}{\partial \rho} dS \\ &= \int_{\partial B_r} \frac{\partial(|x_{n+1}|^{2a-1} u_{p-1})}{\partial \rho} - (2a-1) \frac{|x_{n+1}|^{2a-1} u_{p-1}}{r} dS, \quad \rho = |x|. \end{aligned}$$

In getting the last equality, we have used

$$\frac{\partial |x_{n+1}|^{2a-1}}{\partial \rho} = (2a-1) \frac{|x_{n+1}|^{2a-1}}{\rho}, \quad \rho = |x|.$$

Taking  $w_{p-1}(x) = |x_{n+1}|^{2a-1}u_{p-1}(x)$ , one gets that

$$(r^n \bar{w}'_{p-1} - (2a-1)r^{n-1}\bar{w}_{p-1})' \leq 0.$$

This implies that

$$r^n \bar{w}'_{p-1} - (2a-1)r^{n-1}\bar{w}_{p-1} < 0 \Rightarrow \frac{\partial(r^{-(2a-1)}\bar{w}_{p-1})}{\partial r} < 0, \text{ for } r > 0.$$

Set  $z_{p-1}(r) = r^{-(2a-1)}\bar{w}_{p-1}(r)$ , then

$$z_{p-1}(r) < z_{p-1}(0) < 0, \text{ for all } r > r_1 = 0.$$

$z_{p-1}(0) < 0$  follows from

$$\begin{aligned} z_{p-1}(0) &= \lim_{r \rightarrow 0} r^{-(n+2a-1)} \int_{\partial B_r} |x_{n+1}|^{2a-1} u_{p-1} dS \\ &= \lim_{r \rightarrow 0} r^{-n} \int_{\partial B_r} |\cos \theta_1|^{2a-1} u_{p-1} dS, \quad x_{n+1} = r \cos \theta_1 \\ &= u_{p-1}(0) \int_{\partial B_1} |\cos \theta_1|^{2a-1} dS < 0. \end{aligned}$$

Repeating the above steps, it is easy to see that

$$z'_{p-2}(r) > -c_1 z_{p-1}(0)r.$$

Hence

$$z_{p-2}(r) \geq c_2 r^2, \text{ for } r \geq r_2 > r_1.$$

By induction, it follows that

$$(-1)^i z_{p-i}(r) \geq c_i r^{2(i-1)} \text{ for } r \geq r_i, i = 1, \dots, p, \quad \text{for } c_i > 0.$$

Hence if  $p$  is odd, it's a contradiction with  $u > 0$ .

So  $p$  must be even such that

$$z_0(r) \geq c_0 r^{\sigma_0}, \sigma_0 = 2(p-1)$$

and

$$(-1)^i z_{p-i}, (-1)^i z'_{p-i} > 0, i = 1, \dots, p, \text{ for } r > r_0 > 0.$$

We set  $A = 2\alpha(p-1) + n + 2a + 2p$  and assume that

$$z_0(r) \geq \frac{c_0^{\alpha_k} r^{\sigma_k}}{A^{b_k}}, \forall r \geq r_k.$$

Integrating by parts, one gets

$$\begin{aligned} (3.1) \quad & \int_{\partial B_r} \left( \frac{\partial (|x_{n+1}|^{2a-1} u_{p-1})}{\partial \rho} - (2a-1) \frac{|x_{n+1}|^{2a-1} u_{p-1}}{r} \right) dS \\ & - \int_{\partial B_{r_k}} \left( \frac{\partial (|x_{n+1}|^{2a-1} u_{p-1})}{\partial \rho} - (2a-1) \frac{|x_{n+1}|^{2a-1} u_{p-1}}{r_k} \right) dS + \int_{B_r \setminus B_{r_k}} |x_{n+1}|^{2a-1} u^\alpha dx = 0. \end{aligned}$$

In fact, we have

$$\begin{aligned}
\int_{B_r \setminus B_{r_k}} |x_{n+1}|^{2a-1} u^\alpha dx &= \int_{r_k}^r d\rho \int_{\partial B_\rho} (|x_{n+1}|^{2a-1} u)^\alpha |x_{n+1}|^{(1-\alpha)(2a-1)} dS \\
&\geq \int_{r_k}^r \rho^{n+(1-\alpha)(2a-1)} \bar{w}_0^\alpha d\rho \\
&= \int_{r_k}^r \rho^{n+2a-1} z_0^\alpha d\rho
\end{aligned}$$

In getting the second inequality, we have used  $\alpha > 1$ ,  $|x_{n+1}| < \rho$  and the convexity of  $g(t) = t^\alpha$ . Therefore, (3.1) means that

$$\begin{aligned}
(3.2) \quad r^{n+2a-1} z'_{p-1}(r) &\leq r_k^{n+2a-1} z'_{p-1}(r_k) - \int_{r_k}^r \rho^{n+2a-1} z_0^\alpha d\rho \\
&\leq -c_0^{\alpha^{k+1}} \frac{r^{\alpha\sigma_k+n+2a} - r_k^{\alpha\sigma_k+n+2a}}{A^{\alpha b_k}(\alpha\sigma_k + n + 2a)} \\
&\leq -\frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+n+2a}}{2A^{\alpha b_k}(\alpha\sigma_k + n + 2a)}
\end{aligned}$$

for  $r \geq 2^{\frac{1}{\alpha\sigma_k+n+2a}} r_k$ , since  $z'_{p-1}(r_k) < 0$ . Hence

$$z'_{p-1}(r) \leq -\frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+1}}{2A^{\alpha b_k}(\alpha\sigma_k + n + 2a)}.$$

Then

$$\begin{aligned}
z_{p-1}(r) &\leq -\frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+2}}{4A^{\alpha b_k}(\alpha\sigma_k + n + 2a)(\alpha\sigma_k + 2)} \\
&\leq -\frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+2}}{4A^{\alpha b_k}(\alpha\sigma_k + n + 2a)^2}
\end{aligned}$$

for  $r \geq 2^{\frac{2}{\alpha\sigma_k+1}} r_k$ . By induction, it follows that

$$(-1)^i z_{p-i}(r) \geq \frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+2i}}{A^{\alpha b_k} 4^i (\alpha\sigma_k + n + 2a + 2p)^{2i}}, r \geq 2^{\frac{2i}{\alpha\sigma_k+1}} r_k.$$

Especially,

$$z_0(r) \geq \frac{c_0^{\alpha^{k+1}} r^{\alpha\sigma_k+2p}}{A^{\alpha b_k} 4^p (\alpha\sigma_k + n + 2a + 2p)^{2p}}, r \geq 2^{\frac{2p}{\alpha\sigma_k+1}} r_k.$$

Set  $\sigma_0 = 2(p-1)$ ,  $r_0$ , then

$$\sigma_{k+1} = \alpha\sigma_k + 2p, r_{k+1} = 2^{\frac{2p}{\alpha\sigma_k+1}} r_k.$$

First of all, by mathematical induction, it is easy to see that

$$A^{\alpha b_k} 4^p (\alpha\sigma_k + n + 2a + 2p)^{2p} \leq A^{2p(k+1)+\alpha b_k}$$

if we notice that

$$2(\alpha\sigma_k + n + 2a + 2p) \leq A(\alpha\sigma_{k-1} + n + 2a + 2p).$$

Thus, we also can set

$$b_0 = 0, b_{k+1} = \alpha b_k + 2p(k+1).$$

Then we have

$$z_0(r) \geq \frac{c_0^{\alpha^{k+1}} r^{\sigma_{k+1}}}{A^{b_{k+1}}}, r \geq r_{k+1}.$$

It's very important that we shall notice that

$$r_{k+1} \leq cr_0$$

where  $c$  can be chosen to be  $2^{\sum_{k=0}^{\infty} \frac{2p}{\alpha\sigma_k+1}}$ .

By a direct computation, we have

$$\sigma_k = 2(p-1)\alpha^k + \frac{2p(\alpha^k - 1)}{\alpha - 1}, b_k = 2p \left( \frac{\alpha(\alpha^k - 1)}{(\alpha - 1)^2} - \frac{k}{\alpha - 1} \right).$$

Taking  $\bar{r} = \max(\frac{2A^{\frac{2p\alpha}{(\alpha-1)^2}}}{c_0}, cr_0)$ , we will have

$$z_0(\bar{r}) \geq \frac{c_0^{\alpha^{k+1}} \bar{r}^{\sigma_{k+1}}}{A^{b_{k+1}}} \rightarrow \infty \text{ as } k \rightarrow \infty$$

which is a contradiction. Hence

$$u_{p-1} > 0.$$

Next we claim that

$$u_{p-i} > 0, i = 2, \dots, p-1.$$

By induction, we have for  $i = 1, \dots, k$ ,

$$u_{p-k} > 0.$$

If  $u_{p-k-1}(0) < 0$ , then from

$$-\tilde{\Delta}_{n+1,a} u_{p-k-1} = u_{p-k} > 0,$$

following the same arguments as  $k = 1$ , we have  $z_{p-k-1}(r) < z_{p-k-1}(0) < 0$ . Also

$$(-1)^{i-k} z_{p-i} \geq c_1 r^{2(i-k-1)}, \quad \text{for } r \geq r_0, p \geq i \geq k+1.$$

If  $p-k$  is odd, it is a contradiction to  $z_0 > 0$ . Then  $p-k$  must be even, this means  $z_0(r) \geq cr^{2(p-k-1)} \geq cr^2$  for  $r \geq r_0 > 1$ . By (3.2), one can see

$$\begin{aligned} r^{n+2a-1} z'_{p-1}(r) &\leq r_0^{n+2a-1} z'_{p-1}(r_0) - \int_{r_0}^r \rho^{n+2a-1} z_0^\alpha d\rho \\ (3.3) \quad &\leq r_0^{n+2a-1} z'_{p-1}(r_0) - \frac{c^\alpha (r^{2\alpha+n+2a} - r_0^{2\alpha+n+2a})}{(2\alpha+n+2a)} \\ &\leq -\frac{c^\alpha r^{2\alpha+n+2a}}{2(2\alpha+n+2a)} \end{aligned}$$

for  $r > r_1 \geq r_0$  since  $r_0^{n+2a-1} z'_{p-1}(r_0)$  is bounded. This means

$$z_{p-1}(r) \leq -\frac{c^\alpha r^{2\alpha+2}}{4(n+2a+2\alpha)}$$

for  $r \geq r_2 > r_1$  which contradicts to  $z_{p-1} > 0$ . This ends the proof of Theorem 1.2.



## 4. A DIVERGENCE IDENTITY IN A PUNCTURED DOMAIN

In order to prove Theorem 1.3, we begin with the following lemma.

**Lemma 4.1.** *If all the assumptions of Theorem 1.3 are satisfied, then for  $i = 0, \dots, p$*

$$(4.1) \quad \left| \lim_{s \rightarrow 0} \int_{B_1 \setminus B_s} v_i dx \right| < \infty, \quad \text{here } v_i = |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a})^i u.$$

*Proof.* We only need to verify for  $i = p-1$ . Set  $g(x) = |x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha$ . Then  $g \in L^1(B_1)$  by assumption.

$$\begin{aligned} & \int_{B_1 \setminus B_r} |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a})^p u dx = \int_{B_1 \setminus B_r} g dx \\ \Rightarrow & - \int_{\partial B_1} \left( \frac{\partial v_{p-1}}{\partial \rho} - (2a-1)v_{p-1} \right) dS + \int_{\partial B_r} \left( \frac{\partial v_{p-1}}{\partial \rho} - (2a-1)r^{-1}v_{p-1} \right) dS = |g|_{L^1(B_1)} + o(1) \\ \Rightarrow & r^n \bar{v}'_{p-1} - (2a-1)r^{n-1}\bar{v}_{p-1} = |g|_{L^1(B_1)} + \bar{v}'_{p-1}(1) - (2a-1)\bar{v}_{p-1}(1) + o(1) = \beta_{p-1} + o(1) \\ \Rightarrow & (r^{-(2a-1)}\bar{v}_{p-1})' = r^{-(n+2a-1)}(\beta_{p-1} + o(1)) \\ \Rightarrow & r^{n-1}\bar{v}_{p-1}(r) = -(n+2a-2)^{-1}\beta_{p-1} + o(1) \end{aligned}$$

In the above equations, we always set  $\beta_{p-1} = |g|_{L^1(B_1)} + \bar{v}'_{p-1}(1) - (2a-1)\bar{v}_{p-1}(1)$ . Then we get

$$\left| \int_{B_1 \setminus B_r} v_{p-1} dx \right| = \left| \int_r^1 d\rho \int_{\partial B_\rho} v_{p-1} dS \right| = \left| \int_r^1 \rho^n \bar{v}_{p-1} d\rho \right| < \infty$$

For the other  $v_i$ , by induction and repeating the arguments of  $v_{p-1}$ , we can get the same conclusion.  $\square$

With the aid of Lemma 4.1, we now can prove Theorem 1.3.

**The proof of Theorem 1.3:** Claim:

$$(4.2) \quad \int_{B_1} |x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha \varphi dx = \int_{B_1} |x_{n+1}|^{2a-1} u \varphi_p dx,$$

here  $\varphi_p$  is defined as follows:

$$\varphi_0 = \varphi(|x|), \varphi_i = (-\tilde{\Delta}_{n+1,a})^i \varphi_0, \varphi \in C_c^\infty(B_1), i = 0, \dots, p.$$

Set  $\eta(t) \in C^\infty(\mathbb{R})$  with  $\eta(t) = 0, t \leq 1, \eta(t) = 1, t \geq 2$  and  $\eta_\epsilon(x) = \eta(\frac{|x|}{\epsilon})$ . Taking  $\varphi \eta_\epsilon$  as the test function, we will have

$$\begin{aligned} & \int \varphi \eta_\epsilon |x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha dx \\ = & \int (-\tilde{\Delta}_{n+1,a})^{p-1} u(x) L(\varphi \eta_\epsilon) dx, \quad L(f) = |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a}) f \\ = & \int (-\tilde{\Delta}_{n+1,a})^{p-1} u(x) (\eta_\epsilon L(\varphi) + \varphi L(\eta_\epsilon) + 2|x_{n+1}|^{2a-1} \nabla \varphi \nabla \eta_\epsilon) dx \\ = & \int v_{p-1} (\eta_\epsilon \varphi_1 + \psi_1) dx, \quad \text{as } v_{p-1} = |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a})^{p-1} u \end{aligned}$$

where  $\psi_1 = -\varphi \tilde{\Delta}_{n+1,a} \eta_\epsilon - 2\nabla \varphi \nabla \eta_\epsilon$  and  $\text{supp} \psi_1 \subset B_{2\epsilon} \setminus B_\epsilon, |\psi_1| \leq C_1 \epsilon^{-2}$ . We repeat the above process to get

$$\begin{aligned} & \int \varphi \eta_\epsilon |x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha dx \\ &= \int v_{p-i} (\eta_\epsilon \varphi_i + \psi_i) dx \\ &= \int |x_{n+1}|^{2a-1} u(x) (\eta_\epsilon \varphi_p + \psi_p) dx, \end{aligned}$$

here  $\psi_{i+1} = -\tilde{\Delta}_{n+1,a} \psi_i - \varphi_i \tilde{\Delta}_{n+1,a} \eta_\epsilon - 2\nabla \varphi_i \nabla \eta_\epsilon, i = 1, \dots, p-1$ . By induction, it is easy to see that

$$|\psi_i| \leq C_i \epsilon^{-2i}, \text{supp} \psi_i \subset B_{2\epsilon} \setminus B_\epsilon, i = 1, \dots, p.$$

Also we have

$$\begin{aligned} & \int u(x) |x_{n+1}|^{2a-1} |\psi_p(x)| dx \\ & \leq C \left( \int_{B_1} |x|^{-\tau} |x_{n+1}|^{2a-1} u^\alpha \right)^{\frac{1}{\alpha}} \left( \int |x_{n+1}|^{2a-1} |x|^{\frac{\alpha' \tau}{\alpha}} |\psi_p|^{\alpha'} \right)^{\frac{1}{\alpha'}} \\ & \leq C \epsilon^{\frac{2p}{\alpha}} \left( \int_{B_1} |x|^{-\tau} |x_{n+1}|^{2a-1} u^\alpha \right)^{\frac{1}{\alpha}} \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ . Let  $\epsilon \rightarrow 0$ , the claim (4.2) follows immediately. From the arguments of Lemma 4.1, one can see that

$$\begin{aligned} (4.3) \quad & r^n \bar{v}'_i - (2a-1) r^{n-1} \bar{v}_i = \beta_i + o(1), \\ & r^{n-1} \bar{v}_i(r) = -(n+2a-2)^{-1} \beta_i + o(1), \end{aligned}$$

where  $\beta_i = \lim_{s \rightarrow 0} |v_{i+1}|_{L^1(B_1 \setminus B_s)} + \bar{v}'_i(1) - (2a-1) \bar{v}_i(1)$  which is a finite number. Integrating by parts, we can get

$$\begin{aligned} (4.4) \quad & \int_{B_1 \setminus B_r} |x_{n+1}|^{2a-1} u \varphi_p dx = \int_{B_1 \setminus B_r} \varphi |x_{n+1}|^{2a-1} (-\tilde{\Delta}_{n+1,a})^p u dx \\ & - \sum_{k=0}^{p-1} \int_{\partial B_r} \varphi_k \left( \frac{\partial v_{p-1-k}}{\partial \rho} - (2a-1) r^{-1} v_{p-1-k} \right) + v_{p-1-k} \frac{\partial \varphi_k}{\partial \rho} dS. \end{aligned}$$

By the definition of  $\varphi_k$ , we can see that  $\varphi_k$  is radially symmetric as  $\varphi_0$  is radially symmetric. Thus by the average estimates (4.3) and Lemma 4.1, one can get by taking  $r \rightarrow 0$  in (4.4),

$$\int_{B_1} |x_{n+1}|^{2a-1} |x|^{-\tau} u^\alpha \varphi_0 dx = \int_{B_1} |x_{n+1}|^{2a-1} u \varphi_p dx + \sum_{i=0}^{p-1} \beta_i \varphi_i(0).$$

This means by (4.2),

$$\sum_{i=0}^{p-1} \beta_i \varphi_i(0) = 0.$$

We can choose  $\varphi_0^{(i)}(x) \equiv |x|^{2i}, i = 0, \dots, p-1$  for  $|x| \leq \frac{1}{2}$  one by one. For  $\varphi_0^{(0)}(x)$ , we can easily get  $\beta_0 = 0$ . Then by induction and choosing suitable  $\varphi_0^{(i)}(x)$ , we get  $\beta_i = 0$ . It is easy to see that  $\beta_i = 0$  is equivalent to (1.8) holds if we notice the definition of  $\beta_i$ .

## 5. NONEXISTENCE AND CLASSIFICATION OF POSITIVE SOLUTIONS

Next we give a lemma without proof which is due to [5].

**Lemma 5.1.** *If  $u \in C^{2p}(\mathbb{R}^{n+1})$ ,  $p \geq 1$  is radially symmetric and satisfies the inequalities*

$$(-\tilde{\Delta}_{n+1,a})^k u \geq 0, \text{ in } \mathbb{R}^{n+1}, \quad k = 0, 1, \dots, p$$

where  $2p < n + 2a$ . Then we have

$$(ru' + (n + 2a - 2p)u)' < 0.$$

**Definition 5.1.** *Let  $l$  be a positive number. We say that a  $C^2$  function  $f$  has a harmonic asymptotic expansion at infinity in a neighborhood of infinity if:*

$$(5.1) \quad \begin{aligned} f(x) &= \frac{1}{|x|^l} \left( a_0 + \sum_{i=1}^{n+1} \frac{a_i x_i}{|x|^2} \right) + O\left(\frac{1}{|x|^{l+2}}\right), \\ f_{x_i}(x) &= -la_0 \frac{x_i}{|x|^{l+2}} + O\left(\frac{1}{|x|^{l+2}}\right), \quad i = 1, 2, \dots, n+1, \\ f_{x_i x_j} &= O\left(\frac{1}{|x|^{l+2}}\right), \quad i, j = 1, \dots, n+1, \end{aligned}$$

where  $a_i \in \mathbb{R}$  and  $a_0 > 0$ .

Set

$$\Sigma_\lambda = \{x \in \mathbb{R}^{n+1} | x_1 < \lambda\}, \quad x^\lambda = (2\lambda - x_1, x_2, \dots, x_{n+1}).$$

**Lemma 5.2.** *Let  $f$  be a function in a neighborhood at infinity satisfying the asymptotic expansion (5.1). Then there exist  $\lambda_0 > 0$  and  $R > 0$  such that if  $\lambda \geq \lambda_0$ ,*

$$f(x) > f(x^\lambda), \quad \text{for } x_1 < \lambda, x \notin B_R(0).$$

**Lemma 5.3.** *Let  $f$  be a  $C^2$  positive solution of  $-\tilde{\Delta}_{n+1,a} f = F(x)$  for  $|x| > R$  and  $f, F$  be even functions with respect to  $x_{n+1}$ , where  $f$  has a harmonic asymptotic expansion (5.1) at infinity with  $a_0 > 0$ . Suppose that, for some positive number  $\lambda_0$  and for every  $(x_1, x')$  with  $x_1 < \lambda_0$ ,*

$$f(x_1, x') > f(2\lambda_0 - x_1, x') \quad \text{and} \quad F(x_1, x') \geq F(2\lambda_0 - x_1, x').$$

Then there exist  $\epsilon > 0, S > R$  such that

$$(i) \quad f_{x_1}(x_1, x') < 0 \quad \text{in } |x_1 - \lambda_0| < \epsilon, |x| > S, \\ (ii) \quad f(x_1, x') > f(2\lambda_0 - x_1, x') \quad \text{in } x_1 < \lambda_0 - \frac{1}{2}\epsilon < \lambda, |x| > S,$$

for all  $x \in \Sigma_\lambda, \lambda \geq \lambda_1$  with  $|\lambda_1 - \lambda_0| < c_0 \epsilon$ , where  $c_0$  is a positive number depending on  $\lambda_0$  and  $f$ .

Now if  $u(x, y) \in C^{2p}(\overline{R_+^{n+1}})$  satisfies (1.3), we have  $\tilde{u}(x, x_{n+1}) = u(x, \frac{x_{n+1}^2}{4})$  satisfies (1.5). By Lemma 2.1, one can extend  $\tilde{u}$  to  $\mathbb{R}^{n+1}$  by  $\tilde{u}(x, x_{n+1}) = \tilde{u}(x, -x_{n+1})$ ,  $x_{n+1} < 0$  such that  $\tilde{u}(x, x_{n+1})$  still satisfies (1.5) in  $\mathbb{R}^{n+1}$ . Define Kelvin transformation as follows

$$u^*(x) = |x|^{-(n+2a-2p)} \tilde{u}\left(\frac{x}{|x|^2}\right),$$

then by a direct computation,  $u^*$  satisfies

$$(-\tilde{\Delta}_{n+1,a})^p u^* = |x|^{-\tau} (u^*)^\alpha, \quad \text{in } \mathbb{R}^{n+1}$$

where  $\tau = n + 2a + 2p - \alpha(n + 2a - 2p) \geq 0$ , see Appendix for a derivation.

Set

$$u_i^* = (-\tilde{\Delta}_{n+1,a})^i u^* \left( \frac{x}{|x|^2} \right).$$

Then Lemma 2.5 tells us that  $u_i^*$  has the asymptotic behavior (5.1) at  $\infty$ . Moreover, we have the following lemma.

**Lemma 5.4.**

$$|x_{n+1}|^{2a-1} |x|^{-\tau} (u^*)^\alpha \in L^1(B_1).$$

*Proof.* If not, by noting equation (4.4), we set  $r = 1, r_k = r$  in (4.4). Then for  $r$  small enough, there holds

$$\int_{\partial B_r} \left( \frac{\partial (|x_{n+1}|^{2a-1} u_{p-1}^*)}{\partial \rho} - (2a-1) \frac{|x_{n+1}|^{2a-1} u_{p-1}^*}{r} \right) dS \leq -c_1 \int_{B_1 \setminus B_r} |x_{n+1}|^{2a-1} |x|^{-\tau} (u^*)^\alpha dx.$$

Thus we can get,

$$\frac{\partial z_{p-1}}{\partial r} \geq c_1 r^{-(n+2a-1)} \int_{B_1 \setminus B_r} |x_{n+1}|^{2a-1} |x|^{-\tau} (u^*)^\alpha.$$

Or,

$$z_{p-1} \leq -c_1 r^{-(n+2a-2)}, \text{ for } r \leq r_1.$$

Similarly,

$$(-1)^i z_{p-i} \geq c_i r^{-(n+2a-2i)}, r \leq r_i, i = 1, \dots, p,$$

$z_i(r)$  is defined as  $z_i(r) = r^{-(n+2a-1)} \int_{\partial B_r} |x_{n+1}|^{2a-1} u_i^* dS$ . Thus if  $p$  is odd, this means  $z_0 < 0$  which is a contradiction.

We only need to consider the case  $p$  is even. Then

$$z_1(r) < 0, \text{ for } r < r_{p-1},$$

or,

$$\tilde{\Delta}_{n+1,a} z_0 > 0.$$

This means that  $z'_0(r) < 0$  for  $r$  small otherwise  $z_0$  is increasing for  $r$  small and hence  $z_0(r) \leq C$  which contradicts to  $z_0 \geq cr^{-(n+2a-2p)}$  for  $r$  small.

Set

$$z^*(s) = s^{-(n+2a-2p)} z_0 \left( \frac{1}{s} \right).$$

We shall have

$$(-\tilde{\Delta}_{n+1,a})^p z^*(s) \geq (z^*)^\alpha(s).$$

By the same arguments as in the proof of Theorem 1.2, we have

$$(-\tilde{\Delta}_{n+1,a})^i z^*(s) > 0, i = 1, \dots, p.$$

By Lemma 5.1, this means

$$(s(z^*))'(s) + (n+2a-2p)z^*(s)' < 0.$$

Or,

$$s(z^*)''(s) + (n+2a+1-2p)z^*(s) < 0.$$

Now an easy computation yields that

$$\tilde{\Delta}_{n+1,a} z_0(r) = r^{-(n+2a+4-2p)} \left( (z^*)''(s) + \frac{n+2a-2p+1}{s} (z^*)'(s) \right) + \frac{2p-2}{r} z'_0(r) < 0, r = \frac{1}{s},$$

which yields a contradiction.  $\square$

**Lemma 5.5.**  $u_i^* > 0$ , in  $\mathbb{R}^{n+1} \setminus \{0\}$ .

*Proof.* We need only to verify  $u_{p-1}^* > 0$ , the proofs for the other  $u_i^*$  are the same. If not, we can find  $x_0 \in \mathbb{R}^{n+1} \setminus \{0\}$  such that  $u_{p-1}^*(x_0) = -c_0 < 0$ . As the same arguments in the proof of Theorem 1.2 in Section 3, we may assume that the  $n+1$ -th coordinate of  $x_0$  is 0. Set  $v_i(x) = |x_{n+1}|^{2a-1} u_i^*(x)$ . It is easy to see that

$$\lim_{r \rightarrow 0} \frac{1}{r^{n+2a-1}} \int_{\partial B_r(x_0)} v_{p-1}(x) dS < 0.$$

Following the computation in Section 3 yields that

$$\frac{d}{dr} \left[ \frac{1}{r^{n+2a-1}} \int_{\partial B_r} v_{p-1}(x) dS \right] = \frac{1}{r^{n+2a-1}} \int_{\partial B_r} \left[ \frac{\partial v_{p-1}}{\partial r} - (2a-1)r^{-1}v_{p-1} \right] dS < 0.$$

Set  $w(r, x_0) = \frac{1}{r^{n+2a-1}} \int_{\partial B_r(x_0)} v_{p-1}(x) dS$ . Then we can see  $w(r, x_0) < 0$ , for  $r \in [0, |x_0|]$ . Therefore, by Lemma 4.1 and Theorem 1.3, we can choose  $\epsilon$  small enough such that

$$\int_{B_\rho(x_0)} v_{p-1}(x) dx < 0, \rho = |x_0| + \epsilon.$$

Noting that we have  $u_{p-1}^* > 0$  near infinity, then we have for any  $x \in \mathbb{R}^{n+1}$

$$\frac{1}{r^{n+2a-1}} \int_{\partial B_r(x)} v_{p-1} dS > 0, \text{ for } r > |x|.$$

Now we have

$$\int_{B_\rho(x_0)} v_{p-1}(x) dx = \int_{B_\epsilon(0)} v_{p-1} dx + \int_0^1 dt \int_{\partial B_{t|x_0|+\epsilon}(tx_0)} v_{p-1}(x) dx > 0.$$

This yields a contradiction. □

Now we are in a position to prove Theorem 1.1.

**The proof of Theorem 1.1:**

Set  $w_\lambda(x) = u^*(x) - u^*(x^\lambda)$  in  $\Sigma_\lambda$ . From Lemma 5.5 we get  $u_i^* > 0$ . So by Lemma 5.2 and Lemma 2.4, one gets  $u_i^*(x) \rightarrow 0$  as  $x \rightarrow \infty$  to get

$$(-\tilde{\Delta}_{n+1,a})^{p-i} w_\lambda > 0, i = 1, \dots, p.$$

for all  $\lambda \geq \lambda' \gg 1$ . Set

$$\lambda_0 = \inf\{\lambda > 0 \mid (-\tilde{\Delta}_{n+1,a})^{p-i} w_\mu(x) > 0 \text{ in } \Sigma_\mu \text{ for } \mu \geq \lambda, i = 1, \dots, p\}.$$

We may assume  $\lambda_0 > 0$  and by the definition of  $\lambda$ , we see that

$$(-\tilde{\Delta}_{n+1,a})^i w_{\lambda_0}(x) \geq 0, \quad i = 0, \dots, p-1,$$

By Lemma 2.2, we have either

$$(-\tilde{\Delta}_{n+1,a})^i w_{\lambda_0}(x) = 0, \quad i = 0, \dots, p-1.$$

Or

$$(-\tilde{\Delta}_{n+1,a})^i w_{\lambda_0}(x) > 0, \quad i = 0, \dots, p-1.$$

We need to prove the first situation is true. If not, there exist  $\lambda_n \uparrow \lambda_0$  and  $i_0 \in \{1, \dots, p-1\}$  such that

$$\inf_{x \in \mathbb{R}^2} (-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_n}(x) = \inf_{x \in B_R \setminus B_r} (-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_n}(x) = (-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_n}(x^{\lambda_n}) < 0$$

for some fixed  $R$  large and  $r$  small. This is a conclusion from Lemma 5.3, Lemma 5.5 and Lemma 2.2. There are two cases we should distinguish with:

- (1)  $\lim_{n \rightarrow \infty} x^{\lambda_n} = x^0 \in \Sigma_\lambda$ . Then

$$(-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_0}(x^0) = \lim_{n \rightarrow \infty} (-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_n}(x^{\lambda_n}) \leq 0$$

which is a contradiction to  $(-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_0}(x) > 0$ .

- (2)  $\lim_{n \rightarrow \infty} x^{\lambda_n} = x^0 \in \partial \Sigma_\lambda$ . Then

$$\partial_{x_1}(-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_0}(x^0) = \lim_{n \rightarrow \infty} \partial_{x_1}(-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_n}(x^{\lambda_n}) = 0$$

which is a contradiction to  $\partial_{x_1}(-\tilde{\Delta}_{n+1,a})^{i_0} w_{\lambda_0}(x^0) < 0$ .

If  $\alpha < \frac{n+2a+2p}{n+2a-2p}$ , we have  $\tau > 0$ . To prove the radial symmetry of  $u^*(x)$ , one should take a transformation. Set

$$\tilde{u}^*(x', x_{n+1}, x_{n+2}) = u^*(x', \sqrt{x_{n+1}^2 + x_{n+2}^2}).$$

It follows that,

$$(5.2) \quad (-\Delta_{n+2,a-\frac{1}{2}})^p \tilde{u}^* = |x|^{-\tau} (\tilde{u}^*)^\alpha, \text{ in } \mathbb{R}^{n+2}, \partial_{n+2} \tilde{u}^*(x', x_{n+1}, 0) = 0,$$

here  $\tilde{\Delta}_{n+2,a-\frac{1}{2}} = \sum_{i=1}^{n+2} \partial_{x_i}^2 + \frac{2a-2}{x_{n+2}} \partial_{x_{n+2}}$ . There is a singularity at 0, and hence  $\lambda_0$  must be 0.

Notice that (5.2) is rotationally invariant about  $x', x_{n+1}$ . For  $|x'|^2 + x_{n+1}^2 = |\bar{x}'|^2 + \bar{x}_{n+1}^2$ , we have

$$u^*(x', x_{n+1}) = \tilde{u}^*(x', x_{n+1}, 0) = \tilde{u}^*(\bar{x}', \bar{x}_{n+1}, 0) = u^*(\bar{x}', \bar{x}_{n+1}).$$

This implies that

$$\tilde{u}(x', x_{n+1}) = \tilde{u}(\bar{x}', \bar{x}_{n+1}), \text{ if } |x'|^2 + x_{n+1}^2 = |\bar{x}'|^2 + \bar{x}_{n+1}^2.$$

If we take another transformation such as

$$u_b^*(x) = \frac{1}{|x|^{n+2a-2}} \tilde{u}_b \left( \frac{x}{|x|^2} \right), \text{ here } b_{n+1} = 0,$$

where  $\tilde{u}_b(x) = \tilde{u}(x - b)$ . Repeating the above arguments, similarly we have

$$\tilde{u}(x', x_{n+1}) = \tilde{u}(\bar{x}', \bar{x}_{n+1}), \text{ if } |x' + b'|^2 + x_{n+1}^2 = |\bar{x}' + b'|^2 + \bar{x}_{n+1}^2.$$

In fact,  $b'$  can be chosen arbitrarily, thus  $\tilde{u}$  must be a constant. This means that  $\tilde{u} \equiv 0$ .

Now we consider the case  $\alpha = \frac{n+2a+2}{n+2a-2}$  or  $\tau = 0$ . By the same arguments as we did in the case  $\tau > 0$ , there exists  $\lambda = (\lambda_1, \dots, \lambda_{n+1})$  such that

$$(5.3) \quad \tilde{u}^*(x', x_{n+1}, 0) = u^*(x', x_{n+1}) = u^*(\bar{x}', \bar{x}_{n+1}) = \tilde{u}^*(\bar{x}', \bar{x}_{n+1}, 0),$$

if  $\sum_{i=1}^{n+1} |x_i - \lambda_i|^2 = \sum_{i=1}^{n+1} |\bar{x}_i' - \lambda_i|^2$ . In fact,  $\lambda_{n+1}$  must be 0. Otherwise, it follows that

$$u^*(x', 2\lambda_{n+1} - x_{n+1}) = u^*(x', x_{n+1}) = u^*(x', -x_{n+1}).$$

It shows that for the fixed  $x'$ ,  $u^*$  is periodic with respect to  $x_{n+1}$  with period  $2\lambda_{n+1}$ . This means that  $u^*$  must vanish which is impossible. For  $\lambda' = (\lambda_1, \dots, \lambda_n)$ , we have two cases.

- (1)  $\lambda' = 0$ : since  $\tilde{u}(x) = \frac{1}{|x|^{n+2a-2p}} u^*\left(\frac{x}{|x|^2}\right)$ ,  $\tilde{u}(x)$  is radially symmetric with respect to the origin.
- (2)  $\lambda' \neq 0$ : This means that 0 is not the symmetric center of  $u^*$ ,  $u^*$  must be  $C^2$  at 0. In other words,  $\tilde{u}(x)$  has similar asymptotic behaviors at  $\infty$  as  $u^*(x)$ . This allows us to apply the moving plane method to  $\tilde{u}(x)$  directly to obtain that  $\tilde{u}(x)$  is radially symmetric with respect to some point  $b \in \mathbb{R}^{n+1}$ ,  $b_{n+1} = 0$ .

The above arguments show that  $\tilde{u}(x)$  is radially symmetric with respect to some point  $b \in \{b_{n+1} = 0\}$ . Now we can follow the arguments of Section 3 in [3] and use the conformal invariant property to classify the solutions. This completes the proof of Theorem 1.1.

## 6. APPENDIX

In the Appendix, we will prove (2.6). We borrow the ideas from [1] and [14]. As polynomials are dense in  $C^{2k}(B_1)$  and the operator  $\tilde{\Delta}_{n+1,a}$  is linear and local, we only need to show (2.6) is true for all homogeneous polynomials which are even functions with respect to  $x_{n+1}$ . At first, we need a decomposition for polynomials in  $\mathbb{R}^{n+1}$ . Denote the set of all the polynomials which are even with respect to  $x_{n+1}$  by  $\tilde{\mathcal{P}}$ . Set

$$\begin{aligned}\tilde{\mathcal{P}}_m &= \{p \in \tilde{\mathcal{P}} | p \text{ is a homogeneous polynomial with order } m\}, \\ \tilde{\mathcal{H}}_m &= \{p \in \tilde{\mathcal{P}}_m | \tilde{\Delta}_{n+1,a}p = 0\}.\end{aligned}$$

**Lemma 6.1.** *If  $p \in \tilde{\mathcal{P}}$  with  $\deg(p) = m$ , then there exists some polynomial  $q \in \tilde{\mathcal{P}}$  with  $\deg(q) \leq m - 2$  satisfying that*

$$\tilde{\Delta}_{n+1,a}((1 - |x|^2)q + p) = 0.$$

*Proof.* We now define an operator  $T : W \rightarrow W$  by

$$T(q) = \tilde{\Delta}_{n+1,a}((1 - |x|^2)q), \quad q \in W$$

where  $W$  is the set of all polynomials belong to  $\tilde{\mathcal{P}}$  with order less or equal to  $m - 2$ . First we show  $T$  is injective. If  $T(q) = 0$ , this means that  $(1 - |x|^2)q$  solves

$$\tilde{\Delta}_{n+1,a}((1 - |x|^2)q) = 0, \text{ in } B_1, \quad (1 - |x|^2)q = 0, \text{ on } \partial B_1.$$

By Lemma 2.2, we must have  $q = 0$  which implies  $T$  is injective. Note that  $W$  is a finite dimension vector space. This means  $T$  is also surjective. Hence we have for any  $p \in \tilde{\mathcal{P}}$  with  $\deg(p) = m$ , there exists  $q \in W$  such that

$$\tilde{\Delta}_{n+1,a}((1 - |x|^2)q) = -\tilde{\Delta}_{n+1,a}p.$$

□

By Lemma 6.1, we have for any  $p \in \tilde{\mathcal{P}}_m$  there exists  $q \in \tilde{\mathcal{P}}$  with  $\deg(q) \leq m - 2$  such that

$$(6.1) \quad p = h + |x|^2q - q$$

where  $h \in \tilde{\mathcal{P}}$  and  $\tilde{\Delta}_{n+1,a}h = 0$ . Also from the decomposition we know  $\deg(h) \leq m$ . Taking the homogeneous part of degree  $m$  at both sides, we get

$$p = p_m + |x|^2q_{m-2}, p_m \in \tilde{\mathcal{H}}_m, q_{m-2} \in \tilde{\mathcal{P}}_{m-2}.$$

Repeating the above decomposition, we get for  $p \in \tilde{\mathcal{P}}_m$ ,

$$p = p_m + |x|^2p_{m-2} + |x|^4p_{m-4} + \cdots, p_j \in \tilde{\mathcal{H}}_j.$$

The summation of the above decomposition is finite. By such a decomposition, we only need to show (2.6) is true for  $u(x) = |x|^{t-k}h(x) = |x|^th(\frac{x}{|x|})$  where  $t \in \mathbb{R}$  and  $h \in \tilde{\mathcal{H}}_k$  for some  $k$ .

$$\tilde{\Delta}_{n+1,a}u(x) = [t(t + n + 2a - 2) - k(k + n + 2a - 2)]|x|^{t-2}h\left(\frac{x}{|x|}\right)$$

if we note  $x \cdot \nabla h = kh$ . Applying the above identity for  $m$  times, we get

$$(\tilde{\Delta}_{n+1,a})^m u(x) = A_{m,t} |x|^{t-2m} h\left(\frac{x}{|x|}\right)$$

where

$$A_{m,t} = \prod_{j=0}^{m-1} [(t-2j)(t-2j+n+2a-2) - k(k+n+2a-2)].$$

As for  $|x|^{2m-n-2a-t} h(\frac{x}{|x|})$ , we have

$$(\tilde{\Delta}_{n+1,a})^m \left( |x|^{2m-n-2a-t} h\left(\frac{x}{|x|}\right) \right) = B_{m,t} |x|^{-n-2a-t} h\left(\frac{x}{|x|}\right)$$

where

$$B_{m,t} = \prod_{j=0}^{m-1} [(2m-n-2a-t-2j)(2m-t-2j-2) - k(k+n+2a-2)].$$

It is easy to see that  $A_{m,t} = B_{m,t}$  and this proves (2.6).

#### REFERENCES

- [1] S. Axler, P. Bourdon and R. Wade, Harmonic function theory. Springer, 2001.
- [2] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm.Pure Appl.Math., **42** (1989), 271-297.
- [3] W.-X. Chen, C.-M. Li and B. Ou, Classification of solutions for an integral equation, Comm.Pure Appl.Math., **59** (2006), 330-343.
- [4] J. L. Chern and S. G. Yang, A Divergence-type identity in a punctured domain and its application to a singular polyharmonic problem. Journal of Dynamics and Differential Equations, 2004, 16(3): 587-604.
- [5] P. Clément, R. Manásevich and E. Mitidieri, Positive solutions for a quasilinear system via blow up. Comm. Partial Differential Equations 18 (1993), no. 12, 2071-2106.
- [6] B. Gidas, W. M. Ni and L. Nirenberg, "Symmetry of positive solutions of nonlinear elliptic equations in  $\mathbb{R}^n$ " in Mathematical Analysis and Applications, Part A, ed. L. Nachbin, Adv. Math. Suppl. Stud. 7, Academic Press, New York, 1981, 369-402.
- [7] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., **34** (1981), 525-598.
- [8] Q. Han, J. X. Hong and G. G. Huang, Compactness of Alexandrov-Nirenberg Surfaces. Preprint.
- [9] Huang G G. A Liouville theorem of degenerate elliptic equation and its application. Discrete and Continuous Dynamical Systems-Series A. 2013, 33(10): 4549-4566.
- [10] C.-M. Li, *local asymptotic symmetry of singular solutions to nonlinear elliptic equations*, Invent. Math, **123** (1996), 221-231.
- [11] Y.-Y. Li, Remarks on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc., **6** (2004), 153-180.
- [12] C.-S. Lin, A classification of solutions of a conformally invariant fourth order equation in  $\mathbb{R}^n$ , Comment. Math. Helv., **73** (1998), 206-231.
- [13] J. Q. Liu, Y. X. Guo, Y. J. Zhang. Liouville-type theorems for polyharmonic systems in  $\mathbb{R}^N$ . Journal of Differential Equations, 2006, 225(2): 685-709.
- [14] M. Pavlović, Kelvin-Möbius transform of polyharmonic functions.
- [15] J.-C. Wei and X.-W. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. **313** (1999), 207-228.
- [16] X.-W. Xu, Classification of solutions of certain fourth-order nonlinear elliptic equations in  $\mathbb{R}^4$ , Pacific J. Math., **225** (2006), 361-378.

DEPARTMENT OF MATHEMATICS, INS AND MOE-LSC, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI  
E-mail address: genggenhuang@sjtu.edu.cn

DEPARTMENT OF MATHEMATICS, INS AND MOE-LSC, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI



DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF COLORADO AT BOULDER  
*E-mail address:* Congming.Li@Colorado.EDU